Exercise 68

Two curves are **orthogonal** if their tangent lines are perpendicular at each point of intersection. Show that the given families of curves are **orthogonal trajectories** of each other; that is, every curve in one family is orthogonal to every curve in the other family. Sketch both families of curves on the same axes.

$$y = ax^3, \quad x^2 + 3y^2 = b$$

Solution

Differentiate both sides of the given equations with respect to x.

$$\frac{d}{dx}(y) = \frac{d}{dx}(ax^3) \qquad\qquad \frac{d}{dx}(x^2 + 3y^2) = \frac{d}{dx}(b)$$

Use the chain rule to differentiate $y = y(x)$.

$$\frac{dy}{dx} = 3ax^2 \qquad\qquad 2x + 6y\frac{dy}{dx} = 0$$

Solve each equation for dy/dx.

$$\frac{dy}{dx} = 3ax^2 \qquad\qquad 6y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = 3ax^2 \qquad\qquad \frac{dy}{dx} = -\frac{x}{3y}$$

At any point of intersection $y = ax^3$, so the slopes of the tangent lines are as follows.

$$\frac{dy}{dx} = 3ax^2 \qquad\qquad \frac{dy}{dx} = -\frac{x}{3ax^3}$$

$$\frac{dy}{dx} = 3ax^2 \qquad\qquad \frac{dy}{dx} = -\frac{1}{3ax^2}$$

The slopes are negative reciprocals at the points of intersection; therefore, the familes of curves defined by $y = ax^3$ and $x^2 + 3y^2 = b$ are orthogonal trajectories.

Observe that at all points of intersection the tangent lines to the members of each family of curves are orthogonal.